Table 1 Eigenvalues of cantilever beam

Mode	Exact	Condensation	Perturbation
1	6.80096×10^4	6.80933×10^4	6.80096×10^4
2	2.67360×10^6	2.75677×10^6	2.67364×10^6
3	2.10913×10^7	2.50865×10^7	2.11491×10^7

Table 2 Eigenvalues of tail boom

Mode	Exact	Condensation	Perturbation
1	1.72244×10^3	1.72259×10^3	1.72243×10^3
2	1.79624×10^3	1.79637×10^3	1.79620×10^3
3	8.76679×10^3	8.77111×10^3	8.76695×10^3
4	1.48218×10^4	1.48477×10^4	1.48219×10^4
5	1.51423×10^4	1.51703×10^4	1.51424×10^4
6	6.28409×10^4	6.29845×10^4	6.28423×10^4

Table 2 shows the first six eigenvalues. The static condensation gives excellent eigenvalues for up to six modes. The perturbation equation leads the approximation to the exact solution.

Conclusions

A perturbation method using a correction in transformation is presented to improve the condensation solution. The method proved to be effective and is recommended as a postprocessor of the condensation.

Theoretically, the perturbation solution should approach the exact eigenvalue from above. In practice, however, truncation error in the linear equation causes the approximation to overshoot the exact solution. Inclusion of several of the lowest terms can provide excellent estimations for both the eigenvalue and mode shape changes.

One of the most difficult problems is the rate of convergence in the series expansion. The sequential terms in the infinite series may indicate the convergence characteristics in an eigenvalue approximation.

Some of the nonlinear terms may be included to get more accurate solutions. Rather than iterations with nonlinear perturbation equations, integrated reduction methods such as the hybrid dynamic condensation are preferable.

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Structural Approximate Reanalysis for Topological Modifications of Finite Element Systems

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Introduction

IN a general layout optimization problem, possible modifications can be classified as follows[1]:

1) With deletion of members and joints, both the design variable vector and the number of degrees of freedom (DOFs) are reduced. If only members are to be deleted, the value of associated design variables becomes zero and can be eliminated from the set of variables.

2) With addition of members and joints, both the design variable vector and the number of DOFs are increased. When members are added without addition of joints, the vector of design variables is expanded, but the number of DOFs is unchanged.

3) With modification in the geometry, there is no change in the number of variables or in the number of DOFs. In this case, only the numerical values of the variable are modified.

Previous studies have addressed the described cases of layout modifications.[1–6] For the case of addition of members and joints (case 2), Kirsch and Liu[1] presented an effective method to establish a modified initial design. Even though it is suitable for changes in members of a truss structure, it is difficult to apply this method to other models of finite element systems.

In this Note, a new simple and convenient procedure is developed by introducing and reanalyzing the modified initial stiffness matrix (MISM), and a method for forming the expanded basis vectors is introduced. Using this approach, the MISM is formed directly by using the submatrices of an augmented stiffness matrix. Therefore, it is suitable for changes in a general finite element system. Once the MISM is introduced, an expanded basis vector is formed from the MISM; the combined approximations (CA) are used to give an approximate result of the modified structure. The CA method, used in previous studies for structural optimization,[1,5,6] is suitable for problems with unchanged numbers of design variables and DOFs. Because of the large changes involved in topological modification, third-order approximations (CA3) are used in this study. The calculations are based on results of a single exact analysis. Each subsequent reanalysis involves the solution of only a small system of equations. Thus, the computational effort is significantly reduced. In addition, evaluation of derivatives is not required.

Problem Formulation

Static analysis of the initial structure involves solving of a set of simultaneous equations

$$\boldsymbol{K}_0 \boldsymbol{u}_0 = \boldsymbol{R} \tag{1}$$

where $\boldsymbol{K}_0$ is the initial stiffness matrix and $\boldsymbol{u}_0$ is the initial displacement vector; the elements of the load vector $\boldsymbol{R}$ are assumed to be independent of the design variables. However, the approach presented here can also be used to deal with changes in the load vector. The initial stiffness matrix $\boldsymbol{K}_0$ is symmetric and banded, and its decomposition form is available:

$$\boldsymbol{K}_0 = \boldsymbol{L}_0 \boldsymbol{D}_0 \boldsymbol{L}_0^T \tag{2}$$

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where $\boldsymbol{L}_0$ is a lower triangular matrix and $\boldsymbol{D}_0$ is diagonal. Assembly and factorization of stiffness matrix $\boldsymbol{K}$ constitute a large proportion of the computational effort.

The structural changes will result in a changed stiffness matrix $\boldsymbol{K}$ and a changed solution $\boldsymbol{u}$:

$$\boldsymbol{K}\boldsymbol{u} = (\boldsymbol{K}_0 + \Delta\boldsymbol{K})\boldsymbol{u} = \boldsymbol{R} \tag{3}$$

where $\boldsymbol{K}$ is a size-expanded stiffness matrix and $\Delta\boldsymbol{K}$ is the change in the stiffness matrix.

Matrix $\Delta\boldsymbol{K}$ can be divided into submatrices:

$$\Delta\boldsymbol{K} = \begin{bmatrix} \Delta\boldsymbol{K}_{NN} & \Delta\boldsymbol{K}_{NM} \\ \Delta\boldsymbol{K}_{MN} & \Delta\boldsymbol{K}_{MM} \end{bmatrix} \tag{4}$$

where $\Delta\boldsymbol{K}_{MM}$ is a submatrix of stiffness coefficients of the new added joints for the case of addition of members and joints (case 2) where the number of DOFs is increased, in which subscript N is the number of DOFs of the initial structure and M is the augmentation of the DOFs of the modified structure.

Thus, the stiffness matrix $\boldsymbol{K}$ of the changed structure can be expressed in the following submatrix form:

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}_0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \Delta\boldsymbol{K}_{NN} & \Delta\boldsymbol{K}_{NM} \\ \Delta\boldsymbol{K}_{MN} & \Delta\boldsymbol{K}_{MM} \end{bmatrix} \tag{5}$$

The problem under consideration can be formulated as follows: Given $\boldsymbol{K}_0$ and $\boldsymbol{u}_0$, how are approximate solutions found for the modified displacements $\boldsymbol{u}$ without solving the modified analysis equations (3)?

Once the displacements $\boldsymbol{u}$ are solved, the stresses σ can be calculated by the explicit stress-displacement relations

$$\sigma = \boldsymbol{S}\boldsymbol{u} \tag{6}$$

where $\boldsymbol{S}$ is the stress-transformation matrix.

Reanalysis Method

Topological modifications can be divided into two types with respect to the change of DOFs[1]: 1) the common case, where the number of DOF is not increased, and 2) the more challenging case, considered in this Note, where the number of DOFs is increased.

Establishing a Modified Initial Stiffness Matrix (MISM)

Adding joints to the structure increases the number of DOFs and expands the size of stiffness matrix. Therefore, it is necessary first to expand the basis vectors and to introduce an MISM, so that the new DOFs are included in the analysis model.

Considering a given initial design and adding new joints and members, the change in the stiffness matrix $\Delta\boldsymbol{K}$ is divided as Eq. (4). The modified initial stiffness matrix $\boldsymbol{K}_M$ can be formed as follows:

$$\boldsymbol{K}_M = \begin{bmatrix} \boldsymbol{K}_0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \Delta\boldsymbol{K}_{NM} \\ \Delta\boldsymbol{K}_{MN} & \Delta\boldsymbol{K}_{MM} \end{bmatrix} = \begin{bmatrix} \boldsymbol{K}_0 & \Delta\boldsymbol{K}_{NM} \\ \Delta\boldsymbol{K}_{MN} & \Delta\boldsymbol{K}_{MM} \end{bmatrix} \tag{7}$$

Once the MISM is formed, it is then possible to analyze conveniently modified structures, keeping the number of DOFs unchanged. By this approach, any requested design can be analyzed at a later stage by $\Delta\boldsymbol{K}_{NN}$, the other part submatrix of the $\Delta\boldsymbol{K}$.

The modified initial analysis equations are

$$\boldsymbol{K}_M\boldsymbol{u}_M = \boldsymbol{R}_M \tag{8}$$

If the new joints are not loaded, then the modified load vector can be expressed as

$$\boldsymbol{R}_M = \begin{Bmatrix} \boldsymbol{R} \\ \boldsymbol{0} \end{Bmatrix} \tag{9}$$

Equation (7) describes the MISM. It is included by the original stiffness matrix, which is unaltered and occupies the leading diagonal submatrix partition, and the alteration is included through the submatrices of $\Delta\boldsymbol{K}$, which is written in augmenting rows and columns. Some important properties of the original stiffness matrix (symmetry, banding) are preserved in Eq. (7), but positive definiteness is not. This poses no problem for the factorization described in Eq. (2), but other factorizations, such as Choleski, are not suitable.

The LDL^T factorization produces the factors by processing principal diagonal submatrices. It starts with a 1×1 submatrix, then moves through a 2×2 and a 3×3 submatrix, progressively producing and factoring larger leading diagonal submatrices by adding one row and one column, until the factored submatrix is the complete matrix. With the form of Eq. (7), the initial stiffness matrix is given from the initial analysis. The added rows and columns present no conceptual or algorithmic difficulty; the new factors are found simply by restarting the factorization process as if it had been terminated one equation before. Indeed, that the new row and column need only be formed when the factorization reaches them is the basis of the frontal method of solution developed by Irons[7] for the out-of-core solution of finite element equations. Further applications are given elsewhere.[8]

The modified initial displacement vector is given by

$$\boldsymbol{u}_M = \begin{Bmatrix} \boldsymbol{u}_0 \\ \Delta\boldsymbol{u}_0 \end{Bmatrix} \tag{10}$$

Substituting Eq. (10) into Eq. (8), from the second equation we get

$$\Delta\boldsymbol{K}_{MN}\boldsymbol{u}_0 + \Delta\boldsymbol{K}_{MM}\Delta\boldsymbol{u}_0 = 0 \tag{11}$$

Thus,

$$\Delta\boldsymbol{u}_0 = -\Delta\boldsymbol{K}_{MM}^{-1}\Delta\boldsymbol{K}_{MN}\boldsymbol{u}_0 \tag{12}$$

In general, the $M \ll N$, so that solution of Eq. (12) is a very minor step.

For the case where new joints are loaded, the $\boldsymbol{u}_M$ can be directly obtained from Eq. (8) by forward and backward substitution.

Further Changes

Once the MISM has been introduced, evaluation of the displacements for further changes in the other submatrix $\Delta\boldsymbol{K}_{NN}$ of $\Delta\boldsymbol{K}$ is straightforward. The CA approach presented by Kirsch[5,6] has previously been introduced for problems with an unchanged number of design variables and DOFs. In this procedure, the computed terms of series expansion are used as basis vectors in a reduced basis expansion. Because of the large changes involved in topological modification, four vectors assumed in this study, the approximate displacements $\boldsymbol{u}_a$ are expressed as

$$\boldsymbol{u}_a = y_0\boldsymbol{u}_0 + y_1\boldsymbol{u}_1 + y_2\boldsymbol{u}_2 + y_3\boldsymbol{u}_3 = \boldsymbol{U}_B\boldsymbol{y} \tag{13}$$

where $\boldsymbol{u}_0$, $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, and $\boldsymbol{u}_3$ are the first four terms of the series. The matrix $\boldsymbol{U}_B$ and the vector $\boldsymbol{y}$ of coefficients to be determined are defined as

$$\boldsymbol{U}_B = [\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3], \qquad \boldsymbol{y}^T = \{y_0, y_1, y_2, y_3\} \tag{14}$$

Considering the given initial value of the inverse $\boldsymbol{K}_M^{-1}$ and the initial displacements $\boldsymbol{u}_M$, as discussed earlier, in general it is not necessary to calculate the inverse; we can get the decomposed matrix of $\boldsymbol{K}_M$. Using the binomial series, the terms resulting from $\Delta\boldsymbol{K}_{NN}$ for the modified basis vectors can be determined by

$$\begin{aligned} \boldsymbol{u}_0 &= \boldsymbol{u}_M, & \boldsymbol{u}_1 &= -\boldsymbol{K}_M^{-1}\Delta\boldsymbol{K}_{NN}\boldsymbol{u}_M \\ \boldsymbol{u}_2 &= -\boldsymbol{K}_M^{-1}\Delta\boldsymbol{K}_{NN}\boldsymbol{u}_1, & \boldsymbol{u}_3 &= -\boldsymbol{K}_M^{-1}\Delta\boldsymbol{K}_{NN}\boldsymbol{u}_2 \end{aligned} \tag{15}$$

The calculation of the basis vectors involves only forward and backward substitution if $\boldsymbol{K}_M$ is given in the decomposed form from the modified initial analysis. Thus, the third-order terms can be readily calculated. Given $\boldsymbol{K}_M$ and $\boldsymbol{u}_M$, the following procedure is carried out to evaluate the displacements and the stresses for the change of $\Delta\boldsymbol{K}_{NN}$ in the stiffness matrix.[1]

1) The modified matrix $\boldsymbol{K} = \boldsymbol{K}_M + \Delta\boldsymbol{K}_{NN}$ and the basis vectors $\boldsymbol{u}_0$, $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, and $\boldsymbol{u}_3$ expressed by Eq. (15) are introduced.

Table 1 Displacements ($m \times 10^{-3}$) of a modified plate structure

Node	Exact	CA3	Error, %	Node	Exact	CA3	Error, %
5	−0.124	−0.126	1.61	25	−1.326	−1.327	0.08
6	−0.136	−0.139	2.20	26	−1.311	−1.312	0.08
9	−0.446	−0.454	1.79	29	−0.910	−0.919	0.99
10	−0.478	−0.487	1.88	30	−0.919	−0.927	0.87
13	−0.910	−0.919	0.99	33	−0.446	−0.454	1.79
14	−0.919	−0.927	0.87	34	−0.478	−0.487	1.88
17	−1.326	−1.327	0.08	37	−0.124	−0.126	1.61
18	−1.311	−1.312	0.08	38	−0.136	−0.139	2.20
21	−1.496	−1.494	−0.13	47	−0.090	−0.090	0.00
22	−1.482	−1.478	−0.27	49	−0.090	−0.090	0.00

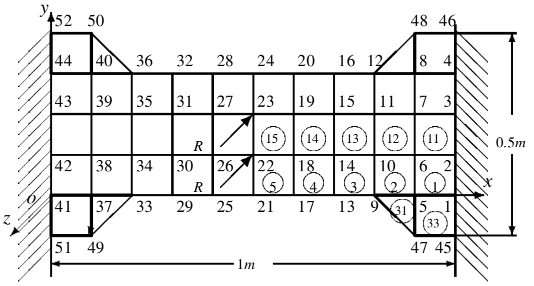

Fig. 1 Topological modifications of plate structure.

2) The reduced (4 × 4) matrix $\boldsymbol{K}_R$ and the reduced (4 × 1) vector $\boldsymbol{R}_R$ can be calculated by the expressions

$$\boldsymbol{K}_R = \boldsymbol{U}_B^T \boldsymbol{K} \boldsymbol{U}_B, \qquad \boldsymbol{R}_R = \boldsymbol{U}_B^T \boldsymbol{R} \tag{16}$$

3) The coefficients vector $\boldsymbol{y}$ are calculated by solving the set of (4 × 4) equations

$$\boldsymbol{K}_R \boldsymbol{y} = \boldsymbol{R}_R \tag{17}$$

4) The final displacements and stresses are evaluated by Eqs. (13) and (6), respectively.

Numerical Example

In the following example, the parameters of the material are the modulus of elasticity $E = 2.1 \times 10^{11}$ Pa and Poisson's ratio $\nu = 0.3$.

Consider an initial design of a rectangle plate structure, shown with thin lines in Fig. 1, where the thickness of the plate is $t = 0.01$ m and the length and width of the plate are 1 and 0.3 m, respectively. The coordination is defined in Fig. 1. The plate is fixed at two sides of the plate. It is discretized into 44 nodes and 30 square plate members, and it is subjected to a single loading condition of two concentrated loads $\boldsymbol{R} = 1000$ N at nodes 22 and 23 along the $-z$ direction.

From the initial analysis, the decomposed form of the initial stiffness matrix $\boldsymbol{K}_0$, the initial load vector $\boldsymbol{R}$, and the initial displacement vector $\boldsymbol{u}_0$ are given.

Consider topological modifications adding new nodes 45–52 and eight members, shown with thick lines in Fig. 1. The change in the stiffness matrix is formed by the finite element program. The MISM and vector $\boldsymbol{u}_M$ are given by Eqs. (7) and (10), respectively, whereas the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2$, and $\boldsymbol{u}_3$ are calculated by Eq. (15). The coefficients vectors $\boldsymbol{y}$ calculated by Eq. (17) are $\{0.940, 0.123, -0.188, -0.039\}^T$.

The resulting displacements are given in Table 1. To illustrate, only the displacements along the z direction are shown. The displacements along the x and y directions are zero. Because of the symmetric property of the structure, only half of the nodes are listed.

Approximate results $\boldsymbol{u}_a$ achieved by CA3 [considering only four basis vectors in Eq. (15)] are summarized in Table 1. Comparing with exact results $\boldsymbol{u}$, the relative errors $(u_i - u_{ai})/u_i \times 100\%$ are also given. The results achieved by the CA3 are very close to the exact solution.

The resulting principal stresses σ_1 of some elements and relative errors are listed in Table 2. The results of the other elements can be obtained from the symmetric property of the structure. The largest errors of the principal stresses in elements 13 and 3 are 12.14% and 9.58%, respectively, and the principal stresses in the other elements have good approximations.

Table 2 Principal stresses (Pa) of a modified plate structure

Element	Exact	CA3	Error, %
1	0.25365E+08	0.25533E+08	−0.66
2	0.17298E+08	0.17774E+08	−2.75
3	0.59322E+07	0.53638E+07	9.58
4	0.14858E+08	0.15120E+08	−1.76
5	0.32834E+08	0.31996E+08	2.55
11	0.26456E+08	0.27132E+08	−2.55
12	0.15933E+08	0.15640E+08	1.84
13	0.27310E+07	0.23994E+07	12.14
14	0.13722E+08	0.13884E+08	−1.18
15	0.32215E+08	0.31359E+08	2.65
31	0.21909E+08	0.22136E+08	−1.04
33	0.19774E+08	0.20084E+08	−1.56

Concluding Remarks

In this Note, a new approximate reanalysis method for topological modifications of general finite element systems has been presented. This presentation is focused on the most challenging case of the addition of joints, in which the structural model and the number of DOFs are changed. From the numerical example, it has been shown that, for changes in topological modifications, the approximate method presented in this Note is effective.

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